

MILNOR INVARIANTS OF LENGTH $2k + 2$ FOR LINKS WITH VANISHING MILNOR INVARIANTS OF LENGTH $\leq k$

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ABSTRACT. J.-B. Meilhan and the second author showed that any Milnor $\overline{\mu}$ -invariant of length between 3 and $2k + 1$ can be represented as a combination of HOMFLYPT polynomial of knots obtained by certain band sum of the link components, if all $\overline{\mu}$ -invariants of length $\leq k$ vanish. They also showed that their formula does not hold for length $2k + 2$. In this paper, we improve their formula to give the $\overline{\mu}$ -invariants of length $2k + 2$ by adding correction terms. The correction terms can be given by a combination of HOMFLYPT polynomial of knots determined by $\overline{\mu}$ -invariants of length $k + 1$. In particular, for any 4-component link the $\overline{\mu}$ -invariants of length 4 are given by our formula, since all $\overline{\mu}$ -invariants of length 1 vanish.

1. INTRODUCTION

For an ordered, oriented link in the 3-sphere, J. Milnor [7, 8] defined a family of invariants, known as *Milnor $\overline{\mu}$ -invariants*. For an n -component link L , Milnor invariant is specified by a sequence I of numbers in $\{1, 2, \dots, n\}$ and denoted by $\overline{\mu}_L(I)$. The length of the sequence I is called the *length* of the Milnor invariant $\overline{\mu}_L(I)$. It is known that Milnor invariants of length two are just linking numbers. In general, Milnor invariant $\overline{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\overline{\mu}_L(J)$ such that J is obtained from I by removing at least one index and permuting the remaining indices cyclicly. If the sequence is non-repeated, then this invariant is also link-homotopy invariant and we call it *Milnor link-homotopy invariant*. Here, the *link-homotopy* is an equivalence relation generated by self-crossing changes.

In [9], M. Polyak gave a formula expressing Milnor invariant of length 3, and in [6], J.-B. Meilhan and the second author generalized it. More precisely, in [6] they showed that any Milnor invariant of length between 3 and $2k + 1$ can be represented as a combination of HOMFLYPT polynomial of knots obtained by certain band sum of the link components, if all Milnor invariants of length $\leq k$ vanish. Their assumption that a link has vanishing Milnor invariants of length $\leq k$ is essential to compute Milnor invariants of length up to $2k + 1$ via their formula. In fact, their formula does not hold for length $2k + 2$ ([6, Section 7]).

In this paper, we improve their formula to give the Milnor invariants of length $2k + 2$ by adding correction terms. Our formula implies that any Milnor invariant of length $2k + 2$ can be given by a combination of HOMFLYPT polynomial of knots obtained by certain band sum operations and knots determined by the first non vanishing Milnor invariants, which are Milnor invariants of length $k + 1$ (Theorem 1.1). In particular, the Milnor invariants of length 4 for any link are given by our formula, since all Milnor invariants of length 1 vanish by the definition (Theorem 1.2).

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Recall that the HOMFLYPT polynomial of a knot K is of the form $P(K; t, z) = \sum_{k=0}^N P_{2k}(K; t)z^{2k}$ and denote by $P_0^{(l)}(K)$ the l -th derivative of $P_0(K; t) \in \mathbb{Z}[t^\pm]$ evaluated at $t = 1$. Denote by $(\log P_0(K))^{(l)}$ the l -th derivative of $\log P_0(K; t)$ evaluated at $t = 1$. We note that $\log P_0(K; t)$ is an additive invariant for knots under the connected sum, since the HOMFLYPT polynomial of knots is multiplicative. In particular, $(\log P_0(K))^{(l)}$ is additive. It is known that $P_0^{(l)}$ is a finite type invariant of degree l [4]. Since $(\log P_0(K))^{(l)}$ is equal to $P_0(K)^{(l)}$ plus a sum of products of $P_0(K)^{(k)}$'s with $k < l$, $(\log P_0)^{(l)}$ is an additive finite type knot invariant of degree l . We also notice that $(\log P_0(K))^{(l)} = P_0^{(l)}(K)$ for $l = 1, 2, 3$, since $P_0(K; 1) = 1$ and $P_0^{(1)}(K; 1) = 0$.

Let $L = \bigcup_{i=1}^n L_i$ be an n -component link in S^3 . Let $I = i_1 i_2 \dots i_m$ be a sequence of m distinct elements of $\{1, 2, \dots, n\}$. Let B_I be an oriented $2m$ -gon, and let p_j ($j = 1, 2, \dots, m$) denote m mutually disjoint edges of ∂B_I according to the boundary orientation. Suppose that B_I is embedded in S^3 such that $B_I \cap L = \bigcup_{j=1}^m p_j$, and such that each p_j is contained in L_{i_j} with opposite orientation. We call such a disk an I -fusion disk for L . For any subsequence J of I , we define the oriented knot L_J as the closure of $\left((\bigcup_{i \in \{J\}} L_i) \cup \partial B_I \right) \setminus \left((\bigcup_{i \in \{J\}} L_i) \cap \partial B_I \right)$, where $\{J\}$ is the subset of $\{1, 2, \dots, n\}$ formed by all indices appearing in the sequence J .

Given a sequence I of elements of $\{1, 2, \dots, n\}$, the notation $J < I$ will be used for any subsequence J of I , possibly empty or equal to I itself, and $|I|$ will denote the length of the sequence I .

Theorem 1.1. *Let L be an n -component link in S^3 ($n \geq 4$) with vanishing Milnor link-homotopy invariants of length $\leq k$. Then for any sequence I of length $2k+2$ of elements of $\{1, 2, \dots, n\}$ without repeated number and for any I -fusion disk for L , we have*

$$\overline{\mu}_L(I) \equiv -\frac{1}{(2k+1)!2^{2k+1}} \sum_{J < I} (-1)^{|J|} (\log P_0(L_J))^{(2k+1)} - \delta_L(I) \pmod{\Delta_L(I)},$$

where $\delta_L(I)$ is an invariant of L that determined by Milnor invariants for length- $(k+1)$ subsequences of I which is defined in Subsection 2.5.

With the same assumption as in Theorem 1.1, the same formula but $\delta_L(I) = 0$ holds for a sequence I with $3 \leq |I| \leq 2k+1$ [6].

We also give the case of 4-component links more clearly.

Theorem 1.2. *Let L be a 4-component link in S^3 . Then for any sequence $I = i_1 i_2 i_3 i_4$ of distinct elements of $\{1, 2, 3, 4\}$ and for any I -fusion disk for L , we have*

$$\overline{\mu}_L(I) \equiv -\frac{1}{48} \sum_{J < I} (-1)^{|J|} P_0^{(3)}(L_J) - \frac{1}{2} x_{i_1 i_3} x_{i_2 i_4} (x_{i_1 i_3} + x_{i_2 i_4} - 1) \pmod{\Delta_L(I)},$$

where x_{ij} is the linking number of i -th component and j -th component of L .

Remark 1.3. We note that x_{ij} is divisible by $\Delta_L(I)$ if ij is a subsequence of I . Hence the correction term $\frac{1}{2} x_{i_1 i_3} x_{i_2 i_4} (x_{i_1 i_3} + x_{i_2 i_4} - 1)$ vanishes up to modulo $\Delta_L(I)$ if either $x_{i_1 i_3}$ or $x_{i_2 i_4}$ is even.

Remark 1.4. We can generalize Theorem 1.1 and Theorem 1.2 about all repeated sequences by the same arguments as those in [6, Introduction]. That is, we have formulae for not only Milnor link-homotopy invariants but also Milnor isotopy invariants.

2. PRELIMINARY

2.1. String link. Let n be a positive integer, and let $D^2 \subset \mathbb{R}^2$ be the unit disk equipped with n marked points x_1, x_2, \dots, x_n in its interior, lying in the diameter on the x -axis of \mathbb{R}^2 . An n -string link (or n -component string link) is the image of a proper embedding $\sqcup_{i=1}^n [0, 1]_i \rightarrow D^2 \times [0, 1]$ of the disjoint union $\sqcup_{i=1}^n [0, 1]_i$ of n copies of $[0, 1]$ in $D^2 \times [0, 1]$, such that for each i the image of $[0, 1]_i$ runs from $(x_i, 0)$ to $(x_i, 1)$. Each string of an n -string link is equipped with an (upward) orientation. The n -string link $\{x_1, x_2, \dots, x_n\} \times [0, 1]$ in $D^2 \times [0, 1]$ is called the *trivial n -string link* and denoted by $\mathbf{1}_n$. Let y_1, y_2, \dots, y_n be points in ∂D^2 in Figure 4, $p_i = x_i y_i$ ($i = 1, 2, \dots, n$) and $q_j = y_j x_{j+1}$ ($j = 1, 2, \dots, n-1$) segments, and q_n an arc in D^2 connecting y_n and x_1 such that $\bigcup_{i=1}^n (p_i \cup q_j)$ bounds the shaded disk in Figure 4. Then for an n -string link l , the knot

$$l \cup \left(\bigcup_{i=1}^n \left((p_i \times \{1\}) \cup (y_i \times [0, 1]) \cup (q_i \times \{0\}) \right) \right)$$

is called the *closure knot* of l . Note that the link

$$l \cup \left(\bigcup_{i=1}^n \left(p_i \times \{0, 1\} \cup y_i \times [0, 1] \right) \right)$$

is the *closure* \hat{l} of l in the usual sense.

The set of isotopy classes of n -string links fixing the endpoints has a monoid structure, with composition given by the stacking product and with the trivial n -string link $\mathbf{1}_n$ as unit element. Given two n -string links L and L' , we denote their product by $L \times L'$, which is obtained by stacking L' above L and reparametrizing the ambient cylinder $D^2 \times [0, 1]$.

2.2. Clasper. Clasper is defined by K. Habiro [2]. Here we define only tree clasper. For a general definition of clasper, we refer the reader to [2].

Let L be a (string) link. A disk T embedded in S^3 (or $D^2 \times [0, 1]$) is called a *tree clasper* for L if it satisfies the following three conditions:

- (1) T is decomposed into disks and bands, called *edges*, each of which connects two distinct disks.
- (2) The disks have either 1 or 3 incident edges. We call a disk with 1 incident edge a *leaf*.
- (3) T intersects L transversely, and the intersections are contained in the union of the interiors of the leaves.

Throughout this paper, the drawing convention for clasps are those of [2, Figure 7], unless otherwise specified.

The *degree* of a tree clasper T is defined as the number of leaves minus 1. A tree clasper of degree k is called a C_k -tree. A tree clasper for a (string) link L is *simple* if each of its leaves intersects L at exactly one point. Let T be a simple tree clasper for an n -component (string) link L . The *index* of T is the collection of all integers i such that T intersects the i -th component of L .

Given a C_k -tree T for a (string) link L , there is a procedure to construct a framed link $\gamma(T)$ in a regular neighborhood of T . *Surgery along* T means surgery along $\gamma(T)$. Since there exists an orientation-preserving homeomorphism, fixing the boundary, from the regular neighborhood $N(T)$ of T to the manifold $N(T)_T$ obtained from $N(T)$ by surgery along T , surgery along T can be regarded as a local move on L . We say that the resulting link L_T is obtained from L by surgery along T . For example, surgery along a simple C_k -tree is a local move as illustrated in Figure 1.

Similarly, for a disjoint union of trees $T_1 \cup \dots \cup T_m$ for L , we can define $L_{T_1 \cup \dots \cup T_m}$ as a link obtained by surgery along $T_1 \cup \dots \cup T_m$. We often regard $L \cup T_1 \cup \dots \cup T_m$ as $L_{T_1 \cup \dots \cup T_m}$.

The C_k -equivalence is an equivalence relation on (string) links generated by surgeries along C_k -tree claspers and isotopies. We use the notation $L \sim_{C_k} L'$ for C_k -equivalent (string) links L and L' .

2.3. Linear trees and planarity. For $k \geq 3$, a C_k -tree T having the shape of the tree clasper in Figure 1 is called a *linear C_k -tree*. The left-most and right-most leaves of T in Figure 1 are called the *ends* of T .

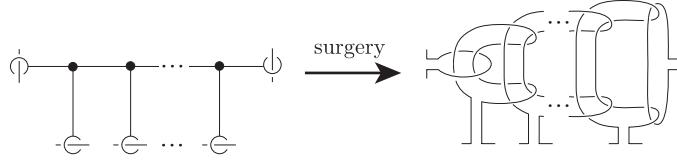


FIGURE 1. Surgery along a simple tree.

Now suppose that T is a linear C_k -tree for some knot K , and denote its ends by f and f' . Then the remaining $k - 1$ leaves of T can be labeled from 1 to $k - 1$, by travelling along the boundary of the disk T from f to f' so that all leaves are visited. We say that T is *planar* if, when traveling along K from f to f' , either following or against the orientation, the labels of the leaves met successively are strictly increasing.

Lemma 2.1. ([6, Lemma 3.2]) *Let T be a non-planar linear tree clasper for a knot K . Then $P_0(K_T; t) = P_0(K; t)$.*

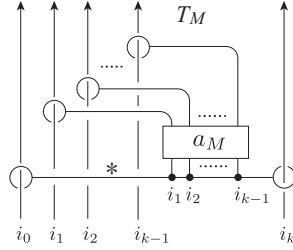
2.4. Presentation of link-homotopy classes for string links. Let \mathcal{M}_k denote the set of all sequences $m_0 m_1 \dots m_k$ of $k + 1$ non-repeating integers from $\{1, 2, \dots, n\}$ such that $m_0 < m_l < m_k$ for $1 \leq l \leq k - 1$. Let $i_0 i_1 \dots i_k$ be a subsequence of $12 \dots n$, and let a_M be a permutation of $\{i_1, i_2, \dots, i_{k-1}\}$. Then $M = i_0 a_M(i_1) \dots a_M(i_{k-1}) i_k$ is in \mathcal{M}_k and all elements of \mathcal{M}_k can be realized in this way. Let T_M be the simple linear C_k -tree for $\mathbf{1}_n$ as illustrated in Figure 2, where $\boxed{a_M}$ is the unique positive k -braid which defined the permutation a_M and such that every pair of strings crosses at most one. In the figure, we also implicitly assume that all edges of T_M overpass all components of $\mathbf{1}_n$. Let T_M^{-1} be the C_k -tree obtained from T_M by inserting a negative half-twist in the $*$ marked edge in Figure 2. We remark that a ‘positive’ half twist is chosen instead of a ‘negative’ one in [6]. Here we choose negative one for a technical reason for the proof of Theorem 1.2.

Denote respectively by V_M and V_M^{-1} the n -string links obtained from $\mathbf{1}_n$ by surgery along T_M and T_M^{-1} . The following theorem is stated in [6] as a slight modified version of Theorem 4.3 in [10].

Theorem 2.2. ([10, Theorem 4.3], [6, Theorem 4.1]) *Any n -string link l is link-homotopic to string link $l_1 \times l_2 \times \dots \times l_{n-1}$, where*

$$l_i = \prod_{M \in \mathcal{M}_i} V_M^{x_M}, \text{ where } x_M = \begin{cases} \mu_l(M) & \text{if } i = 1 \\ \mu_l(M) - \mu_{l_1 l_2 \dots l_{i-1}}(M) & \text{if } i \geq 2. \end{cases}$$

Here, in the product $\prod_{M \in \mathcal{M}_i} V_M^{x_M}$, the string link $V_M^{x_M}$ appears in the lexicographic order of \mathcal{M}_i .

FIGURE 2. The C_k -trees T_M and T_M^{-1}

2.5. The correction term. Let K_M^x and $K_{M,N}^{x,y}$ denote the knot closures of the string links V_M^x and $V_M^x \times V_N^y$ respectively, where x and y are integers, and M and N are subsequences of $12\dots n$.

Let $I = i_1 i_2 \dots i_{2k+2}$ be a sequence of $\{1, 2, \dots, n\}$ without repeated number. Let φ_I be a bijection from $\{1, 2, \dots, 2k+2\}$ to $\{i_1, i_2, \dots, i_{2k+2}\}$ which sends any j to i_j . Let \mathcal{S} be the set of pairs (M, M') such that M and M' are non-successive subsequences of $12\dots(2k+2)$ with length $k+1$, $1 < M$ and $\{M\} \cap \{M'\} = \emptyset$. Then for a link L with vanishing Milnor link-homotopy invariants of length $\leq k$, $\delta_L(I)$ is defined by

$$-\frac{1}{(2k+1)!2^{2k+1}} \sum_{(M, M') \in \mathcal{S}} \left(\log \frac{P_0(K_{M, M'}^{\overline{\mu}_L(\varphi_I(M)), \overline{\mu}_L(\varphi_I(M'))})}{P_0(K_M^{\overline{\mu}_L(\varphi_I(M))}) P_0(K_{M'}^{\overline{\mu}_L(\varphi_I(M'))})} \right)^{(2k+1)},$$

where $\varphi_I(m_1 m_2 \dots m_{k+1})$ means the sequence $\varphi_I(m_1) \varphi_I(m_2) \dots \varphi_I(m_{k+1})$ for a sequence $m_1 m_2 \dots m_{k+1}$. We note that the Milnor invariants of length $k+1$ for L are integer valued invariants and that they are given by linear combinations of $P_0^{(k)}$'s by [6, Theorem 1.2]. We also note that $\delta_L(I)$ is a link-homotopy invariant of L .

Example 2.3. Let $I = i_1 i_2 i_3 i_4$ be a sequence of $\{1, 2, 3, 4\}$ without repeated number. Then \mathcal{S} consists of a single pair $(13, 24)$ of non-successive subsequences of 1234 and

$$\delta_L(I) = -\frac{1}{3!2^3} \left(\log \frac{P_0(K_{13, 24}^{\overline{\mu}_L(i_1 i_3), \overline{\mu}_L(i_2 i_4)})}{P_0(K_{13}^{\overline{\mu}_L(i_1 i_3)}) P_0(K_{24}^{\overline{\mu}_L(i_2 i_4)})} \right)^{(3)}.$$

2.6. Calculus of claspers for parallel claspers. We shall need the following lemma for parallel tree claspers which is given in [6]. For a positive integer m , an m -parallel tree means a family of m parallel copies of a tree clasper.

Lemma 2.4. ([6, Lemma 2.2]) *Let m be a positive integer. Let T be an m -parallel C_k -tree for a (string) link L , and T' be a $C_{k'}$ -tree for L . Here T and T' are disjoint.*

(1) (Leaf slide) *Let $\tilde{T} \cup \widetilde{T'}$ be obtained from $T \cup T'$ by sliding a leaf f' of T' over m parallel leaves of T (see Figure 3 (1)). Then, $L_{T \cup T'}$ is ambient isotopic to $L_{\tilde{T} \cup \widetilde{T'} \cup Y \cup C}$, where Y denotes the m parallel copies of a $C_{k+k'}$ -tree obtained by inserting a vertex v in the edge e of T and connecting v to the edge incident to f' as shown in Figure 3 (1) and where C is a disjoint union of $C_{k+k'+1}$ -trees for L .*

(2) (Edge crossing change) *Let $\tilde{T} \cup \widetilde{T'}$ be obtained from $T \cup T'$ by passing an edge of T' across m parallel edges of T (see Figure 3 (2)). Then, $L_{T \cup T'}$ is ambient isotopic to $L_{\tilde{T} \cup \widetilde{T'} \cup H \cup C'}$, where H denotes the m parallel copies of a $C_{k+k'+1}$ -tree obtained*

by inserting a vertices in both edges, and connecting them by an edge as shown in Figure 3 (2) and where C' is a disjoint union of $C_{k+k'+2}$ -trees for L .

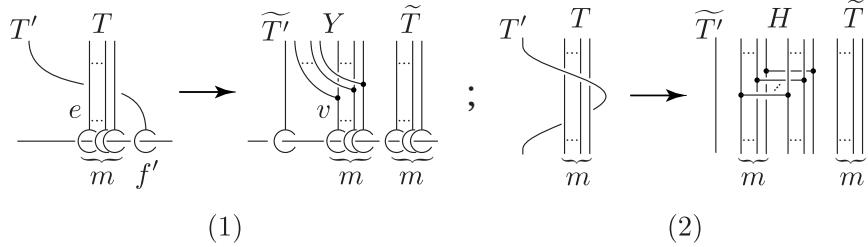


FIGURE 3. Leaf slide and edge crossing change involving parallel trees

Remark 2.5. Leaf slides between C_k -trees for $\mathbf{1}_n$ with the same index can be realized by link-homotopy, since it is realized by surgery along trees intersecting some component of $\mathbf{1}_n$ more than twice and since a surgery along such trees is realized by link-homotopy [1, Lemma 1.2]. Hence, in Subsection 2.4, $V_M^{x_M}$ is link-homotopic to $(\mathbf{1}_n)_{T_M^{x_M}}$, where $T_M^{x_M}$ is $|x_M|$ parallel copies of $T_M^{x_M/|x_M|}$. Note that $V_M^{x_M} = (\mathbf{1}_n)_{T_M^{x_M}}$ if $x_M \geq 0$.

3. PROOF OF THEOREM 1.1

Our strategy of the proof is similar to that in [6, Proof of Theorem 1.1]. In fact, we will use terms ‘good position’ and ‘balanced’ for trees which are defined in [6, Proof of Theorem 1.1] and deform, up to C_n -equivalence, a balanced set of trees with keeping it balanced as well. The big difference is that we have to treat C_k -trees ($n = 2k + 2$) while they did not need to do. We will repeat same arguments as [6, Proof of Theorem 1.1] part way. We remark that a finite type invariant of degree $\leq n - 1$ is an invariant of C_n -equivalence [2], in particular $\log(P_0(K))^{(n-1)}$ is an additive invariant of C_n -equivalence.

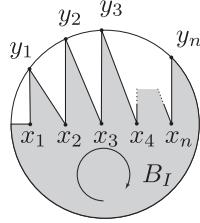
Let $L = \bigcup_{i=1}^n L_i$ be an n -component link in S^3 . Let I be a sequence of $2k + 2$ distinct elements of $\{1, 2, \dots, n\}$. It is sufficient to consider here the case $2k + 2 = n$, because, if $2k + 2 < n$, we have that $\overline{\mu}_L(I) = \overline{\mu}_{\bigcup_{i \in \{I\}} L_i}(I)$. We may further assume that $I = 12\dots n$ without loss of generality. Indeed, for any permutation I' of $12\dots n$, we have that $\overline{\mu}_L(I') = \overline{\mu}_{L'}(12\dots n)$, where L' is obtained from L by reordering the components appropriately.

Let B_I be an I -fusion disk for L . Up to isotopy, we may assume that the $2n$ -gon B_I lies in the unit disk D^2 as shown in Figure 4, where the edges p_j ($j = 1, 2, \dots, n$) are defined by $p_j = x_j y_j$. We may furthermore assume that $L \cup B_I$ lies in the cylinder $D^2 \times [0, 1]$, such that $B_I \subset (D^2 \times \{0\})$, and such that

$$L \cap \partial(D^2 \times [0, 1]) = \bigcup_{j=1}^n \left((p_j \times \{0, 1\}) \cup (\{y_j\} \times [0, 1]) \right).$$

Then, we obtain an n -string link σ whose closure $\hat{\sigma}$ is the link L , by setting

$$\sigma := \overline{L \setminus (L \cap \partial(D^2 \times [0, 1]))}.$$

FIGURE 4. The $2n$ -gon B_I lying in the unit disk D^2 .

For an n -string link $\sigma = \bigcup_{i=1}^n \sigma_i$ and for a subsequence J of $I = 12\dots n$, we denote by $\sigma(J)$ the knot

$$\overline{\left(\left(\bigcup_{j \in \{J\}} \hat{\sigma}_j \right) \cup \partial B_I \right)} \setminus \overline{\left(\left(\bigcup_{j \in \{J\}} \hat{\sigma}_j \right) \cap \partial B_I \right)}.$$

We note that the knot $\sigma(I)$ is equal to the closure knot of σ defined in Subsection 2.1.

Let σ be the n -string link with the closure L defined as above. By combining Theorem 2.2, Remark 2.5, and the assumption that Milnor link-homotopy invariants of length $\leq k$ vanish, σ is link-homotopic to $l_k \times \dots \times l_{2k+1}$, where $l_i = \prod_{M \in \mathcal{M}_i} (\mathbf{1}_n)_{T_M^{x_M}}$. Therefore there is a disjoint union R_1 of simple C_1 -trees whose leaves intersect a single component of $l_k \times \dots \times l_{2k+1}$ such that R_1 is disjoint from $\bigcup_{i=k}^{2k+1} \bigcup_{M \in \mathcal{M}_i} T_M^{x_M}$ and

$$\sigma = (l_k \times \dots \times l_{2k+1})_{R_1}.$$

Set

$$G := \bigcup_{i=k}^{2k+1} \left(\bigcup_{M \in \mathcal{M}_i} T_M^{x_M} \right),$$

then we have

$$\sigma = (\mathbf{1}_n)_{G \cup R_1}.$$

Moreover, by combining Lemmas 4.3 and 4.4 in [6], we have the following:

$$(3.1) \quad \overline{\mu}_L(I) \equiv x_I \pmod{\Delta_L(I)},$$

and

$$(3.2) \quad \Delta_L(I) = \gcd\{x_M \mid M < I, M \neq I\}.$$

A tree for $\mathbf{1}_n$ is said to be in *good position* if each component of $\mathbf{1}_n$ underpasses all edges of the tree. Note that each tree of G is in good position. On the other hand, a tree of R_1 may not be in good position. We now replace R_1 with some trees with good position up to C_n -equivalence. By [2, Proposition 4.5], we have

$$(\mathbf{1}_n)_{G \cup R_1} \sim_{C_n} (\mathbf{1}_n)_{G \cup R},$$

where R is a disjoint union of simple trees for $\mathbf{1}_n$ in good position and intersecting some component of $\mathbf{1}_n$ more than once.

It follows from Lemma 2.1 that for any $J < I$,

$$P_0((\mathbf{1}_n)_{G \cup R}(J)) = P_0((\mathbf{1}_n)_{\tilde{G} \cup R}(J)),$$

where \tilde{G} is obtained from G by eliminating non-planar trees for $(\mathbf{1}_n)(I)$. That is,

$$\tilde{G} = \bigcup_{i=k}^{2k+1} \left(\bigcup_{M \in \mathcal{M}_i, M < I} T_M^{x_M} \right).$$

Here, since $\Delta_L(I)$ divides all x_M with $M < I$ and $M \neq I$ by (3.2), we assume that each $T_M^{x_M}$ is a disjoint union of $\Delta_L(I)$ parallel copies of $T_M^{x_M/|x_M|}$.

We now define the *weight* of a tree t for the trivial knot as a subset of $\{1, 2, \dots, n\}$ and denote it by $w(t)$. A disjoint union $g_1 \cup \dots \cup g_s$ of s trees (possibly parallel) for the trivial knot U is *balanced* if each tree has a weight such that

$$(\mathbf{1}_n)_{\tilde{G} \cup R}(J) \sim_{C_n} U_{(\bigcup_{w(g_i) \subset \{J\}} g_i)},$$

for any $J < I$.

For the knot $(\mathbf{1}_n)_{\tilde{G} \cup R}(I)$, we may think of $\tilde{G} \cup R$ as trees F for the trivial knot $U (= (\mathbf{1}_n)(I))$. We assign the index of each tree of F as weight. Here we recall that the index of a tree for a (string) link is the collection of all integers i such that the tree intersects the i -th component of the (string) link. We may assume that each tree with index $\subset \{J\}$ is also a tree for $(\mathbf{1}_n)(J)$. Then it is obvious that for any $J < I$

$$(\mathbf{1}_n)_{\tilde{G} \cup R}(J) = ((\mathbf{1}_n)(J))_{(\bigcup_{w(g) \subset \{J\}} g)},$$

where $\bigcup_{w(g) \subset \{J\}} g$ means the union of trees g of F with weight $\subset \{J\}$. Since $\tilde{G} \cup R$ is in good position, $((\mathbf{1}_n)(J))_{(\bigcup_{w(g) \subset \{J\}} g)}$ and $U_{(\bigcup_{w(g) \subset \{J\}} g)}$ have a common diagram in $D^2 \times \{0\}$, and hence they are ambient isotopic. In particular F is balanced.

Remark 3.1. When we perform a leaf slide or an edge crossing change between two trees in a balanced union of trees as in Lemma 2.4, we assign the union of weights as weight to each of new trees. More precisely, in Lemma 2.4 (1) (resp. (2)), we assign the weights $w(T)$ and $w(T')$ to \tilde{T} and \tilde{T}' respectively, and assign the union $w(T) \cup w(T')$ to Y (resp. H) and each connected component of C (resp. C'). We note that the union of resulting trees is also balanced.

So far, the proof is the same as [6, Proof of Theorem 1.1]. In [6], they deform F into a balanced union of ‘localized’ tree for the trivial knot U up to C_n -equivalence. But in that case, there are no C_k -trees ($n = 2k+2$). The main difficulty of our proof is how to treat such C_k -trees. In the following, we first deform F into a balanced union of ‘separated’ trees for U except for C_k -trees, and then deform the C_k -trees into suitable shape. Here ‘localized’ implies ‘separated’. We use ‘separated’ instead of ‘localized’, since we notice that we do not need to such a strong condition as ‘localized’. So we also slightly modify [6, Proof of Theorem 1.1] in this sense.

For $M \in \mathcal{M}_i$ ($i = k, \dots, 2k+1$), we denote by $t_M^{\pm 1}$ the tree of F which corresponds to $T_M^{\pm 1}$ of \tilde{G} . Set

$$d := \bigcup_{M \in \mathcal{M}_k, M < I} t_M^{x_M}, \text{ and } D := U_d.$$

Then U_F is obtained from D by surgery along the trees of $F \setminus d$. By using leaf slides and edge crossing changes, we will deform, up to C_n -equivalence, F into a balanced set of ‘separated’ trees for U with fixing d as in claim below.

Claim 3.2. The knot U_F is C_n -equivalent to the connected sum $D \# (\#_{J < I} C^J)$ of knots D and C^J ($J < I$), where C^J is a knot obtained from the trivial knot by surgery along a disjoint union F_J of trees with weight $\{J\}$ and the set $d \cup (\bigcup_{J < I} F_J)$ is balanced. Moreover F_I consists of the parallel tree $t_I^{x_I}$ and $\Delta_L(I)$ -parallel C_{2k+1} -trees.

We define that a tree has *full weight* if the degree of the tree plus one is equal to the number of weight of the tree. We define that a tree is *repeated* if the degree of the tree is more than or equal to the number of weight of the tree.

Proof. We take mutually disjoint 3-balls N_J ($\{J\} \in 2^{\{1, 2, \dots, n\}}$) such that $(N_J, N_J \cap D)$ is a trivial ball-arc pair and $N_J \cap d = \emptyset$. By using leaf slides and edge crossing changes, i.e., by Lemma 2.4 and Remark 3.1, we may assume that all trees except for d with weight $\{J\}$ are contained in the interior of N_J up to C_n -equivalence with keeping the set of trees balanced. Then we have that U_F is C_n -equivalent to $D\#(\#_{J \in I} C^J)$ and C^J is obtained from U by surgery along trees contained in N_J , which are trees with weight $\{J\}$.

To complete the proof, we need to show that the trees F_I in N_I consists of the parallel tree $t_I^{x_I}$ and some $\Delta_L(I)$ -parallel C_{2k+1} -trees. Since t_I is C_{2k+1} -tree and $n = 2k+2$, by Lemma 2.4 we can freely move $t_I^{x_I}$ into N_I up to C_n -equivalence. By Remark 3.1 and the observation below, we see that whenever we apply Lemma 2.4, the new trees we get are repeated or have full weight. Moreover trees have full weight only if they are $\Delta_L(I)$ -parallel trees. Hence we obtain the claim. \square

Observation 3.3. We always move $\Delta_L(I)$ -parallel trees together. If a leaf of new tree obtained by a leaf slide or an edge crossing change interrupts a parallel leaf of a parallel tree, then we sweep the new leaf out of the parallel leaf up to C_n -equivalence. Since the degrees of parallel trees are at least k and the new tree at least $k+1$, we can do such sweeping out easily up to C_n -equivalence by Lemma 2.4.

We consider a leaf slide between a full weight $\Delta_L(I)$ -parallel tree t and a repeated tree t' . Let m be the degree of t and l the degree of t' . If $w(t) \cap w(t') = \emptyset$, then a new C_{m+l} -tree, which is a $\Delta_L(I)$ -parallel tree, has a weight consisting of at most $m+l+1$ elements and new C_{m+l+1} -trees are repeated. If $w(t) \cap w(t') \neq \emptyset$, then all new trees are repeated.

We consider a leaf slide between full weight parallel trees t and t' . We may assume that the degree of t is at least $k+1$ and the degree of t' is at least k . Then the new trees are $\Delta_L(I)$ -parallel trees with degree at least $n-1$.

A leaf slide between repeated trees and an edge crossing change for any case give only repeated trees.

Now we consider D in Claim 3.2. Let \mathcal{S}_k^0 be the set of pairs (M, M') such that M and M' are subsequences of I with length $k+1$, $1 < M$, and $\{M\} \cap \{M'\} = \emptyset$. We also denote by \mathcal{S}_k the subset of \mathcal{S}_k^0 such that both sequences M and M' are not successive. We note that

$$\bigcup_{(M, M') \in \mathcal{S}_k^0} (t_M^{x_M} \cup t_{M'}^{x_{M'}}) = \bigcup_{M \in \mathcal{M}_k, M < I} t_M^{x_M} = d.$$

We separate d into pairwise trees $t_M^{x_M} \cup t_{M'}^{x_{M'}}$ ($(M, M') \in \mathcal{S}_k^0$) by leaf slides and edge crossing changes between different pair of parallel trees. For two parallel trees $t_M^{x_M}$ and $t_N^{x_N}$ which are not pair, we note that $w(t_M) \cap w(t_N) = \{M\} \cap \{N\} \neq \emptyset$. Therefore, when we apply leaf slides or edge crossing changes between $t_M^{x_M}$ and $t_N^{x_N}$, we obtain new trees with degree at least $2k$ which are $\Delta_L(I)$ -parallel trees with full weight and/or repeated trees.

We denote by d_M the parallel tree $t_M^{x_M}$ if $x_M \geq 0$ and the disjoint union of trees obtained from $t_M^{|x_M|}$ by inserting a negative half twist in an edge of each component so that each component of d_M is equal to t_M^{-1} if $x_M < 0$. Then we note that

$$K_M^{x_M} = U_{d_M} \text{ and } K_{M, M'}^{x_M, x_{M'}} = U_{d_M \cup d_{M'}}.$$

By using leaf slides, we have

$$U_{t_M^{x_M}} \sim_{C_{2k}} U_{d_M} \text{ and } U_{t_M^{x_M} \cup t_{M'}^{x_{M'}}} \sim_{C_{2k}} U_{d_M \cup d_{M'}}.$$

where the C_{2k} -equivalence is realized by surgery along repeated C_{2k} -trees. Hence we have that D is C_n -equivalent to the connected sum $D' \# (\#_{(M, M') \in \mathcal{S}_k^0} U_{d_M \cup d_{M'}})$, where D' is obtained from U by surgery along a union of repeated trees. Set

$$\tilde{D} := \#_{(M, M') \in \mathcal{S}_k^0} U_{d_M \cup d_{M'}}, \text{ and } \tilde{d} := \bigcup_{M \in \mathcal{M}_k, M < I} d_M.$$

Hence $D \# (\#_{J < I} C^J)$ is C_n -equivalent to $D' \# \tilde{D} \# (\#_{J < I} \tilde{C}^J)$. By the same reason as Claim 3.2, we have the following claim.

Claim 3.4. The knot U_F is C_n -equivalent to $\tilde{D} \# (\#_{J < I} \tilde{C}^J)$, where \tilde{C}^J is a knot obtained from the trivial knot by surgery along a disjoint union \tilde{F}_J of trees with weight $\{J\}$ and $\tilde{d} \cup (\bigcup_{J < I} \tilde{F}_J)$ is balanced. Moreover \tilde{F}_I consists of the parallel tree $t_I^{x_I}$ and $\Delta_L(I)$ -parallel C_{2k+1} -trees.

Now we have

$$\begin{aligned} \sum_{J < I} (-1)^{|J|} (\log P_0(L_J))^{(n-1)} &= \sum_{J < I} (-1)^{|J|} (\log P_0(\tilde{D}^J \# (\#_{J' < J} \tilde{C}^{J'})))^{(n-1)} \\ &= \sum_{J < I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} \\ &\quad + \sum_{J < I} (-1)^{|J|} \sum_{J' < J} (\log P_0(\tilde{C}^{J'}))^{(n-1)}, \end{aligned}$$

where \tilde{D}^J is a knot obtained from U by surgery along the union of trees in \tilde{d} whose weights are subsets of $\{J\}$. Note that $\tilde{D}^I = \tilde{D} (= \#_{(M, M') \in \mathcal{S}_k^0} K_{M, M'}^{x_M, x_{M'}})$.

For $J' < I$ and $J' \neq I$, the coefficient of $(\log P_0(\tilde{C}^{J'}))^{(n-1)}$ in

$$\sum_{J < I} (-1)^{|J|} \sum_{J' < J} (\log P_0(\tilde{C}^{J'}))^{(n-1)}$$

is equal to 0, since

$$\sum_{J' < J < I} (-1)^{|J|} = \sum_{\{J'\} \subset \{J\} \subset \{I\} \setminus \{a\}} (-1)^{|\{J\}|} + \sum_{\{J'\} \subset \{J\} \subset \{I\} \setminus \{a\}} (-1)^{|\{J\} \cup \{a\}|} = 0,$$

where a is an element in $\{I\} \setminus \{J'\}$. Hence we have

$$\begin{aligned} (3.3) \quad & \sum_{J < I} (-1)^{|J|} (\log P_0(L_J))^{(n-1)} \\ &= \sum_{J < I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} + (\log P_0(\tilde{C}^I))^{(n-1)}. \end{aligned}$$

Note that $(-1)^n = (-1)^{2k+2} = 1$.

On the other hand,

$$\begin{aligned} & \sum_{J < I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} \\ &= \sum_{J < I; J \neq I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} + (\log P_0(\tilde{D}^I))^{(n-1)} \\ &= \sum_{J < I; J \neq I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} + \sum_{(M, M') \in \mathcal{S}_k^0} (\log P_0(K_{M, M'}^{x_M, x_{M'}}))^{(n-1)} \end{aligned}$$

For a subsequence M of I with length $k + 1$, the coefficient of $(\log P_0(K_M^{x_M}))^{(n-1)}$ in $\sum_{J < I; J \neq I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)}$ is

$$\sum_{M < J; J \neq I} (-1)^{|J|} = \sum_{i=0}^k \binom{k+1}{i} \times (-1)^{i+k+1} = -1.$$

This implies that

$$\begin{aligned} & \sum_{J < I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} \\ &= \sum_{(M, M') \in \mathcal{S}_k^0} \left((\log P_0(K_{M, M'}^{x_M, x_{M'}}))^{(n-1)} - (\log P_0(K_M^{x_M}))^{(n-1)} - (\log P_0(K_{M'}^{x_{M'}}))^{(n-1)} \right) \\ &= \sum_{(M, M') \in \mathcal{S}_k^0} \left(\log \frac{P_0(K_{M, M'}^{x_M, x_{M'}})}{P_0(K_M^{x_M}) P_0(K_{M'}^{x_{M'}})} \right)^{(n-1)}. \end{aligned}$$

If $(M, M') \in \mathcal{S}_k^0 \setminus \mathcal{S}_k$, then \tilde{d}_M and $\tilde{d}_{M'}$ are separated by a 2-sphere since either M or M' is a successive sequence. Hence we have

$$K_{M, M'}^{x_M, x_{M'}} = K_M^{x_M} \# K_{M'}^{x_{M'}} \quad ((M, M') \in \mathcal{S}_k^0 \setminus \mathcal{S}_k).$$

It follows that

$$\begin{aligned} (3.4) \quad \sum_{J < I} (-1)^{|J|} (\log P_0(\tilde{D}^J))^{(n-1)} &= \sum_{(M, M') \in \mathcal{S}_k} \left(\log \frac{P_0(K_{M, M'}^{x_M, x_{M'}})}{P_0(K_M^{x_M}) P_0(K_{M'}^{x_{M'}})} \right)^{(n-1)} \\ &= -(n-1)! 2^{n-1} \delta_L(I). \end{aligned}$$

We now consider \tilde{C}^I . Let $h_1^{\Delta_L(I)}, h_2^{\Delta_L(I)}, \dots, h_r^{\Delta_L(I)}$ be the $\Delta_L(I)$ -parallel C_{2k+1} -trees in $\tilde{F}_I \setminus t_I^{x_I}$. Then by using leaf slides and edge crossing changes, we have that

$$\tilde{C}^I \sim_{C_n} (|x_I| \times U_{t_I^{x_I}/|x_I|}) \# (\Delta_L(I) \times (\#_{i=1}^r U_{h_i})),$$

where for positive integer x and for a knot K , $x \times K$ denotes the connected sum of x copies of K . By combining [6, Lemma 3.1 and Claim 5.3 (2)] and (3.1), we have

$$(3.5) \quad -\frac{1}{(n-1)! 2^{n-1}} (\log P_0(\tilde{C}^I))^{(n-1)} \equiv x_I \equiv \bar{\mu}_L(I) \pmod{\Delta_L(I)}.$$

It follows from Equations (3.3), (3.4) and (3.5) that

$$\bar{\mu}_L(I) \equiv -\frac{1}{(n-1)! 2^{n-1}} \sum_{J < I} (-1)^{|J|} (\log P_0(L_J))^{(n-1)} - \delta_L(I) \pmod{\Delta_L(I)}.$$

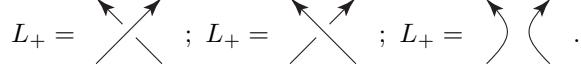
4. PROOF OF THEOREM 1.2

4.1. HOMFLYPT polynomial. First of all, we recall the definition of the HOMFLYPT polynomial, and mention a few useful properties.

The *HOMFLYPT polynomial* $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link L is defined by the following formulas:

- (1) $P(U; t, z) = 1$, and
- (2) $t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z)$,

where U denotes the trivial knot and where L_+ , L_- and L_0 are three links that are identical except in a 3-ball, where they look as follows:



In particular, the HOMFLYPT polynomial of an r -component link L is of the form

$$P(L; t, z) = \sum_{k=1}^N P_{2k-1-r}(L; t) z^{2k-1-r},$$

where $P_{2k-1-r}(L; t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $(2k-1-r)$ -th coefficient polynomial of L . Furthermore, the lowest degree coefficient polynomial of L is given by

$$(4.1) \quad P_{1-r}(L; t) = t^{2Lk(L)} (t^{-1} - t)^{r-1} \prod_{i=1}^r P_0(L_i; t),$$

where L_i is the i -th component of L and $Lk(L)$ is the total linking numbers, see [5, Prop. 22].

4.2. Proof of Theorem 1.2. We may assume that $I = 1234$ by the same reason as those in the proof of Theorem 1.1. By leaf slides and edge crossing changes, we deform the shape of a disjoint union d_M of C_1 -trees which appears in the proof of Theorem 1.1 (here $k = 1$) so that the knot U_{d_M} is as illustrated in Figure 5, which is ambient isotopic to the trivial knot. Since these deformation can be realized by surgery along repeated trees, we obtain Theorem 1.1 for the case when $k = 1$ but different correction term. We remark that the difference of correction terms vanishes modulo $\Delta_L(I)$. Here we have that the new correction term is

$$(\log P_0(K(x_{13}, x_{24})))^{(3)},$$

where $x_{ij} = \overline{\mu}_L(ij)$ and $K(m, n)$ is a knot as illustrated in Figure 6. Since $(\log P_0)^{(3)} = P_0^{(3)}$, we have

$$(4.2) \quad \overline{\mu}_L(I) \equiv -\frac{1}{48} \sum_{J < I} (-1)^{|J|} P_0^{(3)}(L_J) + \frac{1}{48} P_0^{(3)} K(x_{13}, x_{24}) \pmod{\Delta_L(I)}.$$

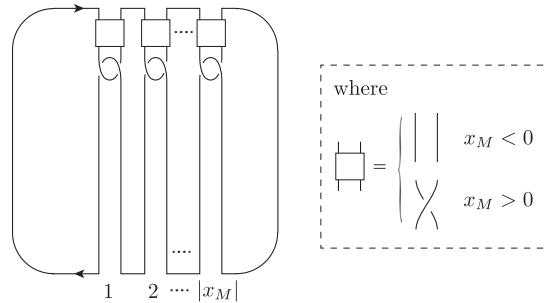
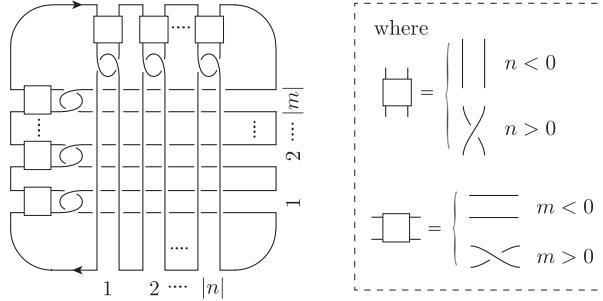


FIGURE 5.

We calculate $P_0(K(m, n))$. Using the relation of the HOMFLYPT polynomial, we obtain the relation

$$(4.3) \quad P_0(K(m, n)) = t^{2\varepsilon} P_0(K(m - \varepsilon, n)) + \varepsilon t^\varepsilon P_{-1}(L(n)),$$

FIGURE 6. $K(m, n)$

where $L(n)$ is illustrated in Figure 7, and $\varepsilon = 1$ (resp. -1) if $m > 0$ (resp. $m < 0$). Since $Lk(L(n)) = n$ and each component of $L(n)$ is trivial, it follows from (4.1) that

$$(4.4) \quad P_{-1}(L(n); t) = t^{2n}(t^{-1} - t).$$

By combining (4.3) and (4.4),

$$P_0(K(m, n)) = t^{2\varepsilon} P_0(K(m - \varepsilon, n)) + \varepsilon t^{2n-1+\varepsilon} - \varepsilon t^{2n+1+\varepsilon}.$$

Since for each $\varepsilon (\in \{-1, 1\})$

$$\varepsilon t^{2n-1+\varepsilon} - \varepsilon t^{2n+1+\varepsilon} = t^{2n} - t^{2n+2\varepsilon},$$

we have

$$P_0(K(m, n)) - t^{2n} = t^{2\varepsilon} (P_0(K(m - \varepsilon, n)) - t^{2n}),$$

and hence

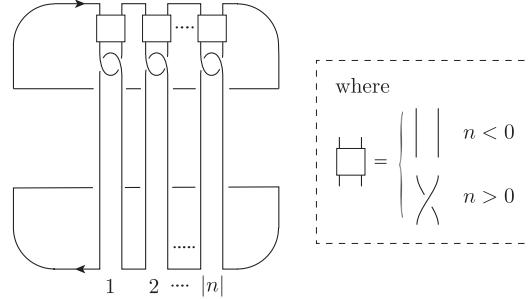
$$P_0(K(m, n)) - t^{2n} = t^{2\varepsilon|m|} (P_0(K(0, n)) - t^{2n}) = t^{2m} (1 - t^{2n}).$$

It follows that we have

$$P_0(K(m, n)) = t^{2m} + t^{2n} - t^{2m+2n},$$

and so we have

$$(4.5) \quad P_0^{(3)}(K(m, n)) = -24mn(m + n - 1).$$

FIGURE 7. $L(n)$

By (4.2) and (4.5) we have the required formula

$$\overline{\mu}_L(I) \equiv -\frac{1}{3!2^3} \sum_{J < I} (-1)^{|J|} P_0^{(3)}(L_J) - \frac{1}{2} x_{13}x_{24}(x_{13} + x_{24} - 1) \pmod{\Delta_L(I)}.$$

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